

SPECTRAL MEASURES ON C-ALGEBRAS OF OPERATORS IN $c_0(\mathbb{N})$

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ABSTRACT. The main goal of this work is to introduce an analogous in the non-archimedean context of the Gelfand spaces of some commutative Banach algebras with unit. In order to do that, we will study the spectrum of this algebras and will show that, under special condition, these algebras are isometrically isomorphic to a respective spaces of continuous functions defined over some compacts. Such isometries will preserve projections and will allow us to define associated measures which are known like spectral measures. We will finish this work by showing that any element of these algebras are integrable under these measures.

1. INTRODUCTION AND NOTATION

Many researchers have tried to generalize the elemental studies of Banach algebras from classical case to vectorial structures over non-archimedean fields. The first big task was to find a results similar to the Gelfand-Mazur Theorem in this context. But, this theorem failed since every field \mathbb{k} with a non-archimedean valuation is contained in another field \mathbb{K} such that its valuation is an extension of the valuation of \mathbb{k} .

One of the main pioneers in the study of non-archimedean Banach algebras of linear operators and spectral theory in this context has been M. Vishik [7], especially in the class of linear operators which admit compact spectrum. We can also mention another important pioneer, Berkovick [3], who made a deep study of this subject on his survey.

The main goal of this work is to introduce an analogous in the non-archimedean context of the Gelfand spaces of some commutative Banach algebras of linear operators with unit. In order to do that, we will study the spectrum of this algebras and will show that, under special condition, these algebras are isometrically isomorphic to a respective spaces of continuous functions defined over some compact. Such isometries will preserve idempotent elements and will allow us to define associated measures which are known as spectral measures. We will finish this work showing that each element of the commutative Banach algebra described before can be represented as an integral of some continuous function, where the integral has been defined by the spectral measure.

Throughout this paper \mathbb{K} is a valued field which is complete with respect to the metric induced by the nontrivial non-archimedean valuation $|\cdot|$ and its residue class field is formally real.

In the classical situation we can distinguish two type of normed spaces: those spaces which are separable and those which are not separable. If E is a separable normed space over \mathbb{K} , then each one-dimensional subspaces of E is homeomorphic to \mathbb{K} , so \mathbb{K} must be separable too. Nevertheless, we know that there exist non-archimedean fields which are not separable. Thus, for non-archimedean normed spaces the concept of separability is meaningless if \mathbb{K} is not separable. However, linearizing the notion of separability, we obtain a useful generalization of this concept. A normed space E over a non-archimedean valued field is said to be of countable type if it contains a countable subset whose linear hull is dense in E . An example of a normed space of countable type is $(c_0, \|\cdot\|_\infty)$, where c_0 is the Banach space of all sequences $x = (a_n)_{n \in \mathbb{N}}$, $a_n \in \mathbb{K}$, for which $\lim_{n \rightarrow \infty} a_n = 0$ and its norm is given by $\|x\|_\infty = \sup \{|a_n| : n \in \mathbb{N}\}$.

A non-archimedean Banach space E is said to be Free Banach space if there exists a family $\{e_i\}_{i \in I}$ of non-null vectors of E such that any element x of E can be written in the form of

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convergent sum $x = \sum_{i \in I} x_i e_i$, $x_i \in \mathbb{K}$, and $\|x\| = \sup_{i \in I} |x_i| \|e_i\|$. The family $\{e_i\}_{i \in I}$ is called orthogonal basis of E . If $s : I \rightarrow (0, \infty)$, then an example of Free Banach space is $c_0(I, \mathbb{K}, s)$, the collection of all $x = (x_i)_{i \in I}$ such that for any $\epsilon > 0$, the set $\{i \in I : |x_i| s(i) > \epsilon\}$ is, at most, finite (or, equivalently, $\lim_{i \in I} |x_i| s(i) = 0$, with respect to the Frechet filter on I) and $\|x\| = \sup_{i \in I} |x_i| s(i)$.

We already know that a Free Banach space E is isometrically isomorphic to $c_0(I, \mathbb{K}, s)$, for some $s : I \rightarrow (0, \infty)$. In particular if a Free Banach space is of countable type, then it is isometrically isomorphic to $c_0(\mathbb{N}, \mathbb{K}, s)$, for some $s : \mathbb{N} \rightarrow (0, \infty)$. Note that if $s(i) \in |\mathbb{K}|$, then E is isometrically isomorphic to $c_0(\mathbb{N}, \mathbb{K})$ (or c_0 in short). For a detailed study of Free Banach spaces, in general, we refer the reader to [4].

Now, since residual class field of \mathbb{K} is formally real, the bilinear form

$$\langle \cdot, \cdot \rangle : c_0 \times c_0 \rightarrow \mathbb{K}; \quad \langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$$

is an inner product, $\|\cdot\| = \sqrt{|\langle \cdot, \cdot \rangle|}$ is a norm in c_0 and the supremum norm $\|\cdot\|_{\infty}$ coincides with $\|\cdot\|$, that is, $\|\cdot\| = \|\cdot\|_{\infty}$ (see [5]). Therefore, to study the Free Banach spaces of countable type it is enough to study the space c_0 .

If E and F are \mathbb{K} -normed spaces, then $\mathcal{L}(E, F)$ will be the \mathbb{K} -normed space consisting of all continuous linear maps from E into F . If $F = E$, then $\mathcal{L}(E) = \mathcal{L}(E, E)$. For any $T \in \mathcal{L}(E, F)$, $N(T)$ will denote its Kernel and $R(T)$ its range.

A linear operator T from E into F is said to be compact operator if $T(B_E)$ is compactoid, where $B_E = \{x \in E : \|x\| \leq 1\}$ is the unit ball of E . It was proved in [6] that T is compact if and only if, for each $\epsilon > 0$, there exists a linear operator of finite-dimensional range S such that $\|T - S\| \leq \epsilon$.

Since c_0 is not orthomodular, there exist operators in $\mathcal{L}(c_0)$ which do not admit adjoint; for example, $T(x) = (\sum_{i=1}^{\infty} x_i) e_1$, $x = (x_i)_{i \in \mathbb{N}} \in c_0$. We will denote by \mathcal{A}_0 the collection of all elements of $\mathcal{L}(c_0)$ which admit adjoint. A characterization of the elements of \mathcal{A}_0 (see [2]) is the following:

$$\mathcal{A}_0 = \left\{ T \in \mathcal{L}(c_0) : \forall y \in c_0, \lim_{i \rightarrow \infty} \langle T e_i, y \rangle = 0 \right\}.$$

Of course, \mathcal{A}_0 is a non-commutative Banach algebra with unit.

Now, for each $a = (a_i)_{i \in \mathbb{N}} \in c_0$, the linear operator M_a , defined by $M_a(\cdot) = \sum_{i=1}^{\infty} a_i \langle \cdot, e_i \rangle e_i$, belongs to \mathcal{A}_0 ; moreover,

$$\lim_{n \rightarrow \infty} \|M_a e_n\| = \lim_{n \rightarrow \infty} \left\| \sum_{i=1}^{\infty} a_i \langle e_n, e_i \rangle e_i \right\| = \lim_{n \rightarrow \infty} |a_n| = 0,$$

meanwhile, the identity map I is also an element of \mathcal{A}_0 , but

$$\lim_{n \rightarrow \infty} \|I(e_n)\| = \lim_{n \rightarrow \infty} \|e_n\| = 1.$$

Let us denote by \mathcal{A}_1 the collection of all $T \in \mathcal{L}(c_0)$ such that $\lim_{n \rightarrow \infty} T e_n = \theta$, i.e.,

$$\mathcal{A}_1 = \left\{ T \in \mathcal{L}(c_0) : \lim_{n \rightarrow \infty} T e_n = \theta \right\}.$$

From the fact that

$$|\langle T e_n, y \rangle| \leq \|T e_n\| \|y\|,$$

we have that $\mathcal{A}_1 \subsetneq \mathcal{A}_0$ since $I \notin \mathcal{A}_1$.

By [4], we know that each $T \in \mathcal{L}(c_0)$ can be represented by $T = \sum_{i,j=1}^{\infty} \alpha_{i,j} e'_j \otimes e_i$, where $\lim_{i \rightarrow \infty} \alpha_{i,j} = 0$, for all $j \in \mathbb{N}$, $\|T\| = \sup \{\|T(e_i)\| : i \in \mathbb{N}\}$ and T is compact if and only if

$$\lim_{j \rightarrow \infty} \sup \{|\alpha_{i,j}| : i \in \mathbb{N}\} = 0.$$

Now, since

$$\begin{aligned}\|Te_n\| &= \left\| \left(\sum_{i,j=1}^{\infty} \alpha_{i,j} e'_j \otimes e_i \right) (e_n) \right\| = \left\| \sum_{i,j=1}^{\infty} \alpha_{i,j} e'_j (e_n) e_i \right\| \\ &= \left\| \sum_{i=1}^{\infty} \alpha_{i,n} e_i \right\| = \sup \{ |\alpha_{i,n}| : i \in \mathbb{N} \}.\end{aligned}$$

Therefore,

$$T \in \mathcal{A}_1 \Leftrightarrow T \in \mathcal{A}_0 \text{ and } T \text{ is compact}$$

Let $\{y^{(i)}\}_{i \in \mathbb{N}}$ be a sequence in c_0 . We will say that $\{y^{(i)}\}_{i \in \mathbb{N}}$ is orthonormal if $\langle y^{(i)}, y^{(j)} \rangle = 0$, $i \neq j$, and $\|y^{(i)}\| = 1$. On the other hand, we will understand by a normal projection to any projection $P : c_0 \rightarrow c_0$ such that $\langle x, y \rangle = 0$ for any $x \in N(P)$ and $y \in R(P)$. For example, if $y \in c_0$, $y \neq 0$, is fixed, then $P(\cdot) = \frac{\langle \cdot, y \rangle}{\langle y, y \rangle} y$ is a normal projection.

The next theorem characterizes compact and self-adjoint operators. Its proof is similar to the proof given in [2], so we will omit it.

Theorem 1. *If the linear operator $T : c_0 \rightarrow c_0$ is compact and self-adjoint, then there exists an element $\lambda = (\lambda_i)_{i \in \mathbb{N}} \in c_0$ and an orthonormal sequence $\{y^{(i)}\}_{i \in \mathbb{N}}$ in c_0 such that*

$$T = \sum_{i=1}^{\infty} \lambda_i P_i$$

and $\|T\| = \|\lambda\|$, where

$$P_i(\cdot) = \frac{\langle \cdot, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)}$$

is the normal projection defined by $y^{(i)}$.

Remark 1. *This theorem gives us a characterization for compact and self-adjoint operators. In fact, it is not hard to see that if we take $\lambda = (\lambda_i)_{i \in \mathbb{N}} \in c_0$ and an orthonormal sequences $\{y^{(i)}\}_{i \in \mathbb{N}}$ in c_0 , the operator*

$$T = \sum_{i=1}^{\infty} \lambda_i P_i,$$

where P_i is as in the Theorem, is compact and self-adjoint and $\|T\| = \|\lambda\|$.

2. ALGEBRA OF OPERATORS

2.1. An algebra without unit. From now on, we will consider a fixed orthonormal sequence $Y = \{y^{(i)}\}_{i \in \mathbb{N}}$ in c_0 . We will denote by $\mathfrak{T}_Y(c_0)$ the collection of all compact operators T_λ , $\lambda \in c_0$, where

$$T_\lambda = \sum_{i=1}^{\infty} \lambda_i P_i$$

As we know, the adjoint T_λ^* of T_λ is itself and $\lim_{n \rightarrow \infty} T_\lambda(e_i) = 0$. Obviously, the collection $\mathfrak{T}_Y(c_0)$ is a linear space, since

$$T_\lambda + T_\mu = T_{\lambda+\mu}; \quad \alpha T_\lambda = T_{\alpha\lambda}$$

On the other hand, since c_0 is a commutative algebra with the operation $\lambda \cdot \mu = (\lambda_i \mu_i)$, we have

$$T_\lambda \circ T_\mu = T_{\lambda \cdot \mu} = T_\mu \circ T_\lambda.$$

Therefore, $\mathfrak{T}_Y(c_0)$ is a commutative algebra without unit. Even more, by the fact that $T_\lambda = T_\mu$ implies $\lambda = \mu$ (see [2]), the linear transformation

$$\Lambda : c_0 \rightarrow \mathfrak{T}_Y(c_0); \quad \lambda \rightarrow \Lambda(\lambda) = T_\lambda$$

is an isometric isomorphism of algebras.

As we know, each algebra without unit can be transformed in an algebra with unit, considering the collection $E^+ = \mathbb{K} \oplus E$ provided with the usual linear operations and the multiplication operation defined by

$$(\alpha, \mu) \cdot (\beta, \nu) = (\alpha\beta, \alpha\nu + \beta\mu + \mu \cdot \nu)$$

The unit of this algebra is $(1, \theta)$. If E is, in particular, a normed space, then E^+ so is and

$$\|(\alpha, \mu)\| = \max\{|\alpha|, \|\mu\|\}.$$

Now, the commutative Banach algebra $(\mathfrak{T}_Y(c_0), +, \cdot, \circ, \|\cdot\|)$ can be transformed, as above, in a commutative Banach algebra $(\mathfrak{T}_Y(c_0)^+, +, \cdot, \circ, \|\cdot\|)$ with unit. By the fact that c_0 is isometrically isomorphic to $\mathfrak{T}_Y(c_0)$, $\mathfrak{T}_Y(c_0)^+$ is isometrically isomorphic to c_0^+ .

2.2. An algebra with unit. We will denote by $\mathcal{S}_Y(c_0)$ the collection of all linear operators $\alpha I + T_\lambda$, $\alpha \in \mathbb{K}$ and $T_\lambda \in \mathfrak{T}_Y(c_0)$. $\mathcal{S}_Y(c_0)$ is a normed space and since

$$\begin{aligned} (\alpha_1 I + T_\mu) \circ (\alpha_2 I + T_\nu) &= \alpha_1 \alpha_2 I + \alpha_1 T_\nu + \alpha_2 T_\mu + T_\mu \circ T_\nu \\ &= \alpha_1 \alpha_2 I + T_{\alpha_1 \nu + \alpha_2 \mu + \mu \nu} \end{aligned}$$

we conclude that $\mathcal{S}_Y(c_0)$ is a commutative algebra with unit.

Theorem 2. *The algebra $\mathcal{S}_Y(c_0)$ is isometrically isomorphic to $\mathfrak{T}_Y(c_0)^+$. As a consequence, $\mathcal{S}_Y(c_0)$ is a commutative Banach algebra with unit.*

Proof. We define

$$\begin{aligned} \mathfrak{T}_Y(c_0)^+ &\rightarrow \mathcal{S}_Y(c_0) \\ (\alpha, T_\lambda) &\rightarrow \alpha I + T_\lambda \end{aligned}$$

Since $\alpha T_\mu + \beta T_\lambda + T_\lambda \circ T_\mu = T_{\alpha\mu + \beta\lambda + \mu\lambda}$, this transformation is an algebra homomorphism. Obviously, this homomorphism is onto; hence it is enough to prove that it is an isometry. We claim that

$$\|\alpha I + T_\lambda\| = \|(\alpha, T_\lambda)\| = \max\{|\alpha|, \|T_\lambda\|\}$$

If $\alpha = 0$ or $\|T_\lambda\| = 0$ or $\|\alpha I\| \neq \|T_\lambda\|$, we are done. We only need to check when

$$|\alpha| = \|\alpha I\| = \|T_\lambda\| \neq 0.$$

Of course,

$$\|\alpha I + T_\lambda\| \leq \max\{|\alpha|, \|T_\lambda\|\}.$$

Now, by the compactness of T_λ ,

$$\lim_{n \rightarrow \infty} T_\lambda(e_n) = 0.$$

Thus, there exists $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow \|T_\lambda(e_n)\| < |\alpha|$$

Therefore,

$$\begin{aligned} \|\alpha I + T_\lambda\| &= \sup\{\|\alpha e_n + T_\lambda(e_n)\| : n \in \mathbb{N}\} \\ &= \max\{\|\alpha e_1 + T_\lambda(e_1)\|, \|\alpha e_2 + T_\lambda(e_2)\|, \dots, \|\alpha e_{N-1} + T_\lambda(e_{N-1})\|, |\alpha|\} \\ &= |\alpha| = \max\{|\alpha|, \|T_\lambda\|\} = \|(\alpha, T_\lambda)\|. \end{aligned}$$

□

Remark 2. *Since c_0^+ is isometrically isomorphic to $\mathfrak{T}_Y(c_0)^+$ and, at the same time, this last is isometrically isomorphic to $\mathcal{S}_Y(c_0)$, c_0^+ is isometrically isomorphic to $\mathcal{S}_Y(c_0)$.*

We claim that the usual norm in $\mathcal{S}_Y(c_0)$ is power multiplicative, that is, for each $T \in \mathcal{S}_Y(c_0)$,

$$\|T^n\| = \|T\|^n.$$

In fact, by the remark, it is enough to study this property in c_0^+ .

The norm in c_0^+ was defined by $\|(\alpha, a)\| = \max\{|\alpha|, \|a\|\}$. On the other hand, for any $(\alpha, a) \in c_0^+$

$$\begin{aligned} (\alpha, a)^2 &= (\alpha^2, 2\alpha a + a^2) \\ (\alpha, a)^3 &= (\alpha^2, 2\alpha a + a^2) \cdot (\alpha, a) \\ &= \left(\alpha^3, \binom{3}{1}\alpha^2 a + \binom{3}{2}\alpha a^2 + \binom{3}{3}a^3\right) \end{aligned}$$

In general, for each $k \in \mathbb{N}$,

$$(\alpha, a)^k = \left(\alpha^k, \sum_{i=1}^k \binom{k}{i} \alpha^{k-i} a^i\right)$$

Now, let us start analyzing $\|(\alpha, a)^k\|$. In general, we have

$$\|(\alpha, a)^k\| \leq \|(\alpha, a)\|^k$$

The power multiplicative property is, obviously, satisfied by the supremum norm in c_0 , that is,

$$\|a^k\| = \|a\|^k.$$

If $\alpha = 0$ or $a = \theta$, then

$$\|(\alpha, a)^k\| = \|(\alpha, a)\|^k$$

If $|\alpha| < \|a\|$, then we have

$$\begin{aligned} \left\| \sum_{i=1}^{k-1} \binom{k}{i} \alpha^{k-i} a^i \right\| &\leq \max \left\{ |\alpha|^{k-i} \|a^i\| : i = 1, \dots, k-1 \right\} \\ &< \max \left\{ \|a\|^{k-i} \|a\|^i : i = 1, \dots, k-1 \right\} = \|a\|^k \end{aligned}$$

Thus,

$$\left\| \sum_{i=1}^k \binom{k}{i} \alpha^{k-i} a^i \right\| = \left\| a^k + \sum_{i=1}^{k-1} \binom{k}{i} \alpha^{k-i} a^i \right\| = \|a\|^k$$

Therefore,

$$\begin{aligned} \|(\alpha, a)\|^k &= \max \left\{ |\alpha|^k, \|a\|^k \right\} \\ &= \|(\alpha, a)^k\| \end{aligned}$$

Suppose, finally, that $|\alpha| \geq \|a\|$. Then,

$$\left\| \sum_{i=1}^k \binom{k}{i} \alpha^{k-i} a^i \right\| \leq \max \left\{ |\alpha|^{k-i} \|a^i\| : i = 1, \dots, k \right\} \leq |\alpha|^k$$

which implies

$$\begin{aligned} \|(\alpha, a)^k\| &= \max \left\{ |\alpha|^k, \left\| \sum_{i=1}^k \binom{k}{i} \alpha^{k-i} a^i \right\| \right\} \\ &= |\alpha|^k = \max \left\{ |\alpha|^k, \|a\|^k \right\} = \|(\alpha, a)\|^k \end{aligned}$$

Definition 1. *An commutative Banach algebra \mathcal{A} with unit is called a C -algebra if there exists a locally compact space X such that \mathcal{A} is isometrically isomorphic to $C_\infty(X)$, where $C_\infty(X)$ is the space of all continuous functions from X into \mathbb{K} which vanishes at infinity.*

As we know $\{e_j = (\delta_{i,j})_{i \in \mathbb{N}} : j \in \mathbb{N}\}$, where $\delta_{i,j}$ denotes the Kronecker symbol, is the canonical basis of c_0 . Since

$$e_j^2 = e_j$$

and

$$\overline{\{e_j : j \in \mathbb{N}\}} = c_0$$

we conclude that the collection of all the idempotent elements of c_0 with norm less than 1 is dense in c_0 . As a consequence, c_0 is a C -algebra (Th. 6.12 [6]).

Theorem 3. $\mathcal{S}_Y(c_0)$ is a C -algebra.

Proof. It follows from the fact that c_0^+ is isometrically isomorphic to $\mathcal{S}_Y(c_0)$. □

Remark 3. *We recall that the spectrum of a commutative Banach algebra \mathfrak{A} is the collection $Sp(\mathfrak{A})$ of all non-null homomorphisms defined from \mathfrak{A} into \mathbb{K} , that is,*

$$Sp(\mathfrak{A}) = \{\phi : \mathfrak{A} \rightarrow \mathbb{K} : \phi \text{ is a non-null homomorphism}\}.$$

Note that the natural topology in $Sp(\mathfrak{A})$ is induced by the product topology in $\mathbb{K}^{\mathfrak{A}}$ and also for each $\phi \in Sp(\mathfrak{A})$, $\|\phi\| \leq 1$. For any $x \in \mathfrak{A}$, we define

$$G_x : Sp(\mathfrak{A}) \rightarrow \mathbb{K}, \quad \phi \rightarrow G_x(\phi) = \phi(x).$$

which is clearly continuous and bounded. Let us denote by

$$\|x\|_{sp} = \sup \{|\phi(x)| : \phi \in Sp(\mathfrak{A})\} = \|G_x\|_\infty$$

the spectral norm of x . Since

$$|\phi(x)| \leq \|\phi\| \|x\| \leq \|x\|,$$

we have, in general, that

$$\|x\|_{sp} \leq \|x\|.$$

Finally, let us denote by $R(G_x)$ the range of G_x . The closure of $R(G_x)$, $\overline{R(G_x)}$, is called spectrum of x .

L. Narici (Cor. 6.16, [6]) proved the following result:

Proposition 1. *A commutative Banach algebra \mathfrak{A} with unit is a C -algebra if and only if its spectrum $Sp(\mathfrak{A})$ is compact and its spectral norm $\|x\|_{sp}$ is equal to $\|x\|$, for every $x \in \mathfrak{A}$.*

Remark 4. *As a consequence of this corollary, we have that $Sp(\mathcal{S}_Y(c_0))$ is compact and, for every $S \in \mathcal{S}_Y(c_0)$,*

$$\|S\| = \sup_{i \in \mathbb{N}} \|S(e_i)\| = \|S\|_{sp}.$$

The compactness of $Sp(\mathcal{S}_Y(c_0))$ guarantees that the spectrum of S is compact in \mathbb{K} , since $\mathcal{S}_Y(c_0)$ has a unit.

3. THE SUBALGEBRA \mathcal{L}_T

Let us fix a compact and self-adjoint operator T ; hence $I + T \in \mathcal{S}_Y(c_0)$. We shall denote by \mathcal{L}_T the closure of the algebra spanned by $\{I, T\}$, that is,

$$\mathcal{L}_T = \overline{\langle \{I, T\} \rangle} = \left\{ \sum_{n=0}^{\infty} \alpha_n T^n : \|\alpha_n T^n\| \rightarrow 0 \right\}.$$

Clearly, \mathcal{L}_T is a C-algebra since it is closed Banach subalgebra of $\mathcal{S}_Y(c_0)$ (Cor. 6.13, [6]).

This condition guarantees that $Sp(\mathcal{L}_T)$ is compact and, for each $H \in \mathcal{L}_T$,

$$\|H\| = \sup_{i \in I} \|H(e_i)\| = \|H\|_{sp}.$$

On the other hand, since the operators norm in $\mathcal{S}_Y(c_0)$ is power multiplicative, such property is inherited by \mathcal{L}_T . Thus, for any $H \in \mathcal{L}_T$, we have

$$\|H^n\| = \|H\|^n.$$

Under the conditions that \mathcal{L}_T is a C-algebra and $Sp(\mathcal{L}_T)$ is compact, we conclude that \mathcal{L}_T is isometrically isomorphic to the space of all continuous functions $C(Sp(\mathcal{L}_T))$ provided by the supremum norm, that is, there exists an isomorphism of algebras Ψ which is, at the same time, an isometry from \mathcal{L}_T onto $C(Sp(\mathcal{L}_T))$. Thus, if $H = \sum_{n=0}^{\infty} \alpha_n T^n$, then $\|H\| = \|\Psi(H)\|_{\infty}$.

Now, since T is compact and self-adjoint, there exists $\lambda = (\lambda_i)_{i \in \mathbb{N}} \in c_0$ for which

$$T = T_{\lambda} = \sum_{n=1}^{\infty} \lambda_i P_i, \quad \|T\| = \|\lambda\| \quad \text{and} \quad T(y^{(i)}) = \lambda_i y^{(i)} \quad \text{for } y^{(i)} \in Y$$

Let us denote by $\sigma(T) = \{\lambda_0, \lambda_1, \dots\}$ with $\lambda_0 = 0$, the collection of all eigenvalues of T . We define the homomorphism of algebra $\phi_i : \langle \{I, T\} \rangle \rightarrow \mathbb{K}$ by

$$\phi_i(T) = \lambda_i.$$

Let $H = \sum_{n=0}^k \alpha_n T_{\lambda}^n \in \langle \{I, T\} \rangle$. Since

$$\begin{aligned} |\phi_i(H)| &= \left| \alpha_0 + \sum_{n=1}^k \alpha_n \lambda_i^n \right| \leq \max \left\{ |\alpha_0|, \left| \sum_{n=1}^k \alpha_n \lambda_i^n \right| \right\} \\ &\leq \max \left\{ |\alpha_0|, \left\| \sum_{n=1}^k \alpha_n \lambda^n \right\| \right\} = \left\| \left(\alpha_0, \sum_{n=1}^k \alpha_n \lambda^n \right) \right\|_{c_0^+} \\ &= \left\| \alpha_0 I + T_{\sum_{n=1}^k \alpha_n \lambda^n} \right\| = \left\| \sum_{n=0}^k \alpha_n T_{\lambda}^n \right\| = \|H\| \end{aligned}$$

we get that ϕ_i is continuous.

Since $\{I, T, T^2, \dots\}$ also generates the closed algebra \mathcal{L}_T and for $H = \sum_{n=0}^{\infty} \alpha_n T^n \in \mathcal{L}_T$, we have

$$\begin{aligned} |\alpha_n \lambda_i^n| &= |\alpha_n| |\lambda_i|^n \leq |\alpha_n| \|\lambda\|^n \\ &= |\alpha_n| \|T\|^n = |\alpha_n| \|T^n\| = \|\alpha_n T^n\| \rightarrow 0, \end{aligned}$$

and then ϕ_i can be continuously extended to \mathcal{L}_T .

From this, the following function

$$\sigma(T) \xrightarrow{\Gamma} Sp(\mathcal{L}_T); \quad \lambda_i \rightarrow \Gamma(\lambda_i) = \phi_i.$$

is well defined. We claim that Γ is bijective. Clearly, it is an injective function. To prove the surjective condition, we will start with the following result:

Theorem 4. *If $z \notin \sigma(T)$, then $zI - T$ is invertible in $\mathcal{S}_Y(c_0)$.*

Proof. For $y \in R(zI - T)$, there exists $x \in c_0$ such that

$$(zI - T)(x) = y$$

Since $z \notin \sigma(T)$, we can solve this equation for x and get

$$(3.1) \quad x = \frac{1}{z}y + \frac{1}{z}Tx = \frac{1}{z}y + \frac{1}{z} \sum_{i=1}^{\infty} \lambda_i \frac{\langle x, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)}$$

Applying the continuous functional $\langle \cdot, y^{(k)} \rangle$ above, we have

$$\begin{aligned} \langle x, y^{(k)} \rangle &= \left\langle \frac{1}{z}y + \frac{1}{z} \sum_{i=1}^{\infty} \lambda_i \frac{\langle x, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)}, y^{(k)} \right\rangle \\ &= \frac{1}{z} \langle y, y^{(k)} \rangle + \frac{1}{z} \frac{\lambda_k}{z} \langle x, y^{(k)} \rangle \end{aligned}$$

Now, solving the last equation for $\langle x, y^{(k)} \rangle$, we obtain

$$\begin{aligned} \left(1 - \frac{\lambda_k}{z}\right) \langle x, y^{(k)} \rangle &= \frac{1}{z} \langle y, y^{(k)} \rangle \\ \langle x, y^{(k)} \rangle &= \frac{1}{z - \lambda_k} \langle y, y^{(k)} \rangle \end{aligned}$$

Note that the sequence

$$\left(\frac{\lambda_k}{z - \lambda_k} \right)_{k \in \mathbb{N}}$$

is an element of c_0 . In fact, since $z \notin \sigma(T)$, we have that $|z| > 0$ and, therefore, for a given $0 < \epsilon < 1$, there exists $i_0 \in \mathbb{N}$ such that

$$i \geq i_0 \Rightarrow |\lambda_i| < \epsilon |z|.$$

Thus,

$$i \geq i_0 \Rightarrow \left| \frac{\lambda_i}{z - \lambda_i} \right| = \frac{|\lambda_i|}{|z|} < \epsilon.$$

Now, replacing in (3.1), we get

$$\begin{aligned} x &= \frac{1}{z}y + \frac{1}{z} \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} \frac{\langle y, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)} \\ &= \frac{1}{z}y + \frac{1}{z} \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(y) \end{aligned}$$

Although y belongs to $R(zI - T)$, the last expression holds for any $y \in c_0$. Thus, if we denote by

$$R_z(T)(y) = \frac{1}{z}y + \frac{1}{z} \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(y),$$

then $R_z(T)(\cdot) \in \mathcal{S}_Y(c_0)$, since $\sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(\cdot)$ is compact and self-adjoint operator. Let us show that, effectively, $R_z(T)(\cdot)$ is the inverse operator of $zI - T$:

$$\begin{aligned} [(zI - T) \circ R_z(T)](y) &= (zI - T) \left(\frac{1}{z} y + \frac{1}{z} \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(y) \right) \\ &= y + \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(y) - \frac{1}{z} \sum_{i=1}^{\infty} \lambda_i P_i(y) - \frac{1}{z} \sum_{i=1}^{\infty} \frac{\lambda_i^2}{z - \lambda_i} P_i(y); \quad T(P_i(y)) = \lambda_i P_i(y) \\ &= y + \sum_{i=1}^{\infty} \left[\underbrace{\frac{\lambda_i}{z - \lambda_i} - \frac{\lambda_i}{z} - \frac{\lambda_i^2}{z(z - \lambda_i)}}_{=0} \right] P_i(y) = y = I(y) \end{aligned}$$

In the other direction, since $P_j(P_i(x)) = P_i(x)$, if $j = i$, otherwise it is 0, we have

$$\begin{aligned} [R_z(T) \circ (zI - T)](x) &= zR_z(T)(x) - R_z(T)(Tx) \\ &= x + \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(x) - \sum_{i=1}^{\infty} \lambda_i R_z(T)(P_i(x)) \\ &= x + \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(x) - \sum_{i=1}^{\infty} \lambda_i \left[\frac{1}{z} P_i(x) + \frac{1}{z} \sum_{j=1}^{\infty} \frac{\lambda_j}{z - \lambda_j} P_j(P_i(x)) \right] \\ &= x + \sum_{i=1}^{\infty} \left[\underbrace{\frac{\lambda_i}{z - \lambda_i} - \frac{\lambda_i}{z} - \frac{\lambda_i^2}{z(z - \lambda_i)}}_{=0} \right] P_i(x) = x = I(x) \end{aligned}$$

Therefore, $R_z(T) = (zI - T)^{-1} \in \mathcal{S}_Y(c_0)$. \square

Corollary 1. *If $z \notin \sigma(T)$, then $R_z(T) \in \mathcal{L}_T$.*

Proof. We already know that \mathcal{L}_T is a C-algebra with unit; hence, by Th. 6.10 in [6], we have that

$$(zI - T)^{-1} \in \overline{\mathbb{K}[zI - T]} = \overline{\left\langle zI - T, (zI - T)^2, (zI - T)^3, \dots \right\rangle}$$

Now, since $(zI - T)^n \in \mathcal{L}_T$ for any $n \in \mathbb{N}$, we have that $\overline{\mathbb{K}[zI - T]}$ is a subalgebra of \mathcal{L}_T . This proves that $R_z(T) \in \mathcal{L}_T$. \square

Corollary 2. *The function $\sigma(T) \xrightarrow{\Gamma} Sp(\mathcal{L}_T)$ is bijective.*

Proof. We already know that Γ is injective. If $\phi \in Sp(\mathcal{L}_T)$, then $\phi(T) = z$, for a $z \in \mathbb{K}$. Suppose that $z \notin \sigma(T)$, hence $zI - T$ has an inverse and, for the previous theorem, $(zI - T)^{-1} = R_z(T) \in \mathcal{L}_T$. Since the function ϕ is a homomorphism of algebras with unit, we have

$$1 = \phi(I) = \phi\left((zI - T)^{-1} \circ (zI - T)\right) = \phi\left((zI - T)^{-1}\right) \phi(zI - T),$$

but, by the linearity of ϕ , the factor $\phi(zI - T)$ is null, which is a contradiction. Such contradiction is coming from the fact that $z \notin \sigma(T)$.

Thus, if $\phi \in Sp(\mathcal{L}_T)$, then there exists $\mu \in \sigma(T)$ such that $\phi = \phi_\mu$ and therefore Γ is bijective. \square

Remark 5. *We already have identified $Sp(\mathcal{L}_T)$ with $\sigma(T)$ through the bijective function Γ . Let us consider the subspace topology of \mathbb{K} on $\sigma(T)$. Also, note that $\sigma(T)$ is compact with the subspace topology.*

Now, we affirm that $\Upsilon = \Gamma^{-1}$ is continuous. In fact, if $\phi_\alpha \rightarrow \phi$ in the induced topology by the product topology in $\mathbb{K}^{\mathcal{L}_T}$, then

$$\phi_\alpha(H) \rightarrow \phi(H)$$

for all $H \in \mathcal{L}_T$, in particular,

$$\phi_\alpha(T) \rightarrow \phi(T)$$

or, equivalently,

$$\Upsilon(\phi_\alpha) \rightarrow \Upsilon(\phi)$$

Now, since Υ is bijective and continuous, $Sp(\mathcal{L}_T)$ is compact and $\sigma(T)$ is a Hausdorff space, we conclude that Υ is a homeomorphism.

By these facts and by the uniqueness of X (up to homeomorphism) for which $\mathcal{L}_T \cong C(X)$, we have

$$\mathcal{L}_T \cong C(\sigma(T)).$$

From the fact that $Sp(\mathcal{L}_T)$ is homeomorphic to $\sigma(T)$,

$$G_T : \sigma(T) \rightarrow \mathbb{K}; \lambda_i \rightarrow G_T(\lambda_i) = \lambda_i$$

in other words, $G_T = f_T$ is the identity map. Thus, if $H = \alpha_0 I + \sum_{n=1}^{\infty} \alpha_n T^n \in \mathcal{L}_T$, then

$$G_H : \sigma(T) \rightarrow \mathbb{K}; \lambda_i \rightarrow G_H(\lambda_i) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \lambda_i^n.$$

Therefore, we can get the well-known Gelfand transformation

$$G : \mathcal{L}_T \rightarrow C(\sigma(T)); H \rightarrow G_H$$

Proposition 2. *G is an isometric isomorphism algebra.*

Proof. Clearly, G is an algebra homomorphism. Since \mathcal{L}_T satisfies the condition given by Cor. 6.16, pp. 218, [6], we have

$$\|G_H\|_\infty = \|H\|_{sp} = \|H\|.$$

Thus, it is enough to prove that G is surjective. By the fact that G is an algebra homomorphism and the image of T by G is the identity map $G_T = f_T$, the collection $\{1, f_T, f_T^2, \dots\}$ is the image of $\{I, T, T^2, \dots\}$.

Now, since $\sigma(T)$ is compact, Kaplanski (Th. 5.28, pp. 191 [6]) guarantees that $[\{1, f_T, f_T^2, \dots\}]$ is dense in $C(\sigma(T))$. Thus, if $f \in C(\sigma(T))$, then there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$ in $[\{1, f_T, f_T^2, \dots\}]$ such that $f = \lim_{n \rightarrow \infty} g_n$. Now, for each $n \in \mathbb{N}$, there exists $H_n \in \mathcal{L}_T$ such that

$$G_{H_n} = g_n.$$

By the fact that G is an isometry, the sequence (H_n) is a Cauchy sequence in \mathcal{L}_T . Let us denote by $H = \lim_{n \rightarrow \infty} H_n$. Since G is continuous, we have that

$$G_H = \lim_{n \rightarrow \infty} G_{H_n} = \lim_{n \rightarrow \infty} g_n = f.$$

□

4. AN INTEGRAL.

By the previous section, there exists an algebra isometric isomorphism $\Psi = G^{-1} : C(\sigma(T)) \rightarrow \mathcal{L}_T$. Let us denote by $\Omega(\sigma(T))$ the Boolean ring of all clopen subsets of $\sigma(T)$.

For a $C \subset \sigma(T)$, the function η_C denotes the characteristic function of C . If $C_1, C_2 \subset \sigma(T)$, then

$$\begin{aligned} \eta_{C_1} \cdot \eta_{C_2} &= \eta_{C_1 \cap C_2}; \quad \eta_C^2 = \eta_C \\ \eta_{C_1} + \eta_{C_2} &= \eta_{C_1 \cup C_2}, \text{ if } C_1 \cap C_2 = \emptyset. \end{aligned}$$

Of course, η_C is continuous if, and only if, $C \in \Omega(\sigma(T))$.

Now, since Ψ is a homomorphism of algebras, we have

$$\Psi(\eta_C) = \Psi(\eta_C^2) = \Psi(\eta_C)^2.$$

In other words, $\Psi(\eta_C)$ is a projection, even more, if $C \in \Omega(\sigma(T)) \setminus \{\emptyset\}$, then $\Psi(\eta_C)$ is a non-null projection in \mathcal{L}_T .

On the other hand, we know that the linear hull of $\{\eta_C : C \in \Omega(\sigma(T))\}$ is dense in $C(\sigma(T))$; hence, for a given $\epsilon > 0$ and for $f \in C(\sigma(T))$, there exists a finite clopen partition $\{C_k : k = 1, \dots, n\}$ of $\sigma(T)$ and a finite collection of scalars $\{\alpha_k : k = 1, \dots, n\}$ such that

$$(4.1) \quad \left\| f - \sum_{k=1}^n \alpha_k \eta_{C_k} \right\|_{\infty} = \sup_{x \in \sigma(T)} \left| f(x) - \sum_{k=1}^n \alpha_k \eta_{C_k}(x) \right| < \epsilon$$

Without loss of generality, we can assume that, for each $k = 1, \dots, n$, there exists $x_k \in \sigma(T)$ such that

$$\left\| f - \sum_{k=1}^n \alpha_k \eta_{C_k} \right\|_{\infty} = \sup_{x \in \sigma(T)} \left| f(x) - \sum_{k=1}^n f(x_k) \eta_{C_k}(x) \right| < \epsilon$$

Using the isometry of Ψ , we have

$$\left\| \Psi(f) - \sum_{k=1}^n f(x_k) \Psi(\eta_{C_k}) \right\| < \epsilon$$

If we denote by E_{C_k} the corresponding projection $\Psi(\eta_{C_k})$, then

$$\left\| \Psi(f) - \sum_{k=1}^n f(x_k) E_{C_k} \right\| < \epsilon$$

This also shows that the space

$$\langle \{E \in \mathcal{L}_T : E^2 = E\} \rangle$$

is dense in \mathcal{L}_T .

Note that these subsets can be classified as follow: the first type (or type 1) are those finite subsets of $\{\lambda_n : n \geq 1\}$ and the second type (or type 2) are those subsets of the form $\sigma(T) \setminus C$ where C is a finite subset of $\{\lambda_n : n \geq 1\}$.

Let us consider the following set-function:

$$m_T : \Omega(\sigma(T)) \rightarrow \mathcal{L}_T; \quad C \rightarrow m_T(C) = \Psi(\eta_C) = E_C.$$

Note that if $C = \{\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_n}\} \in \Omega(\sigma(T))$, then

$$m_T(C) = \sum_{k=1}^n E_{i_k}, \text{ where } E_{i_k} = E_{\{\lambda_{i_k}\}}$$

or, if $C = \sigma(T) \setminus \{\lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_n}\}$, then

$$m_T(C) = I - \sum_{k=1}^n E_{i_k}.$$

On the other hand, if $\mathcal{C} = \{C_\mu\}_{\mu \in I}$ is shrinking of $\Omega(\sigma(T))$ and $\cap_{\mu \in I} C_\mu = \emptyset$, then there exists $\mu_0 \in I$ such that for $\mu \geq \mu_0$, $C_\mu = \emptyset$.

Therefore, we can prove that m_T satisfies:

- (1) If $\{C_k : k = 1, \dots, n\} \subset \Omega(\sigma(T))$ such that $C_h \cap C_k = \emptyset$ for $h \neq k$, then

$$m_T(\cup_{k=1}^n C_k) = \sum_{k=1}^n m_T(C_k).$$

- (2) For all $C \in \Omega(\sigma(T))$, $\{m_T(B) : B \in \Omega(\sigma(T)), B \subset C\}$ is bounded.

- (3) If \mathcal{C} is a collection in $\Omega(\sigma(T))$ is shrinking and $\cap_{C \in \mathcal{C}} C = \emptyset$, then $\lim_{C \in \mathcal{C}} m_T(C) = \theta$.

that is, m_T is a finite additive measure.

Take a $C \in \Omega(\sigma(T))$, $C \neq \emptyset$ and denote by \mathcal{D}_C the collection of all $\alpha = \{C_1, C_2, \dots, C_n; x_1, x_2, \dots, x_n\}$, where $\{C_k : k = 1, \dots, n\}$ is a clopen partition of C and $x_k \in C_k$. We define an order by $\alpha_1 \geq \alpha_2$ if, and only if, the clopen partition of C in α_1 is a refinement of the clopen partition of C in α_2 . Thus, \mathcal{D}_C , with this order, is a directed set.

Now, if $f \in C(\sigma(T))$ and $\alpha = \{C_1, C_2, \dots, C_n; x_1, x_2, \dots, x_n\} \in \mathcal{D}_C$, then we define

$$\omega_\alpha(f, m_T, C) = \sum_{k=1}^n f(x_k) m_T(C_k) = \sum_{k=1}^n f(x_k) E_{C_k}.$$

For each $f \in C(\sigma(T))$ and $C \in \Omega(\sigma(T))$, the continuous function $f\eta_C$ can be reached by a net in $C(\sigma(T))$ of type

$$\left\{ \sum_{k=1}^n f(x_k) \eta_{C_k} \right\}_{\alpha \in \mathcal{D}_C}$$

By the isometry of Ψ ,

$$\lim_{\alpha \in \mathcal{D}_C} \omega_\alpha(f, m_T, C) = \Psi(f\eta_C)$$

Therefore, the operator $\Psi(f)$ can be interpreted as an integral which we will denote by

$$\Psi(f\eta_C) = \int_{\sigma(T)} f\eta_C dm_T = \int_C f dm_T = \lim_{\alpha \in \mathcal{D}_C} \omega_\alpha(f, m_T, C)$$

As a particular cases are $T = \Psi(f_T)$ and $I = \Psi(1)$

$$T = \Psi(f_T) = \int_{\sigma(T)} f_T dm_T; \quad I = \Psi(1) = \int_{\sigma(T)} dm_T.$$

5. FINITE RANGE SELF-ADJOINT OPERATORS

In this section we will suppose that there exists an $n \in \mathbb{N}$ such that $\lambda_k = 0$, for $k \geq n+1$. Of course, the operator Sea $T = \sum_{i=1}^{\infty} \lambda_i P_i = \sum_{i=1}^n \lambda_i P_i$ and it is an operator of finite range.

Note that, in this case, $\sigma(T) = \{\lambda_0, \lambda_1, \dots, \lambda_n\}$ with $\lambda_0 = 0$

As before,

$$\mathcal{L}_T = \left\{ \alpha_0 I + \sum_{j=1}^{\infty} \alpha_j T^j \in \mathcal{L}(c_0) : \lim_{j \rightarrow \infty} \alpha_j T^j = 0 \right\}$$

Since $T^j = \sum_{i=1}^n \lambda_i^j P_i$ and

$$\sum_{j=1}^{\infty} \alpha_j T^j = \sum_{j=1}^{\infty} \sum_{i=1}^n \alpha_j \lambda_i^j P_i = \sum_{i=1}^n \left(\sum_{j=1}^{\infty} \alpha_j \lambda_i^j \right) P_i,$$

we conclude that

$$\mathcal{L}_T \subset [I, P_1, \dots, P_n]$$

Theorem 5. *Suppose that $\lambda_i \neq \lambda_j \forall i, j = 1, 2, \dots, n$ with $i \neq j$. Then,*

$$\mathcal{L}_T = [\{I, P_1, \dots, P_n\}]$$

Proof. Using the Van der Monde method to calculate determinants whose columns are geometric progressions, we have

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1^n & \lambda_2^n & \dots & \lambda_n^n \end{vmatrix} = \left(\prod_{i=1}^n \lambda_i \right) \left(\prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \right) \neq 0$$

since $\lambda_i \neq \lambda_j \forall i \neq j$.

Considering the equation system

$$\begin{aligned} T &= \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_n P_n \\ T^2 &= \lambda_1^2 P_1 + \lambda_2^2 P_2 + \dots + \lambda_n^2 P_n \\ &\dots \\ T^n &= \lambda_1^n P_1 + \lambda_2^n P_2 + \dots + \lambda_n^n P_n \end{aligned}$$

and applying the Van der Monde method for operators, we obtain

$$P_k = \prod_{i=0, i \neq k}^n \left(\frac{\lambda_i I - T}{\lambda_i - \lambda_k} \right) \quad k = 1, 2, \dots, n.$$

Therefore, $P_1, \dots, P_n \in \mathcal{L}_T$ and, as a consequences, $\mathcal{L}_T = \{I, P_1, P_2, \dots, P_n\}$. \square

Remark 6. *On the other hand, suppose, for instance, that $\lambda_1 = \lambda_2$ and $\lambda_i \neq \lambda_j$ for $i, j = 2, \dots, n$ with $i \neq j$. Then,*

$$T = \lambda_1 (P_1 + P_2) + \lambda_3 P_3 + \dots + \lambda_n P_n.$$

Using the same arguments as in the previous theorem, we get that

$$\mathcal{L}_T = [I, P_1 + P_2, P_3, \dots, P_n].$$

Note that $P_1 \notin \mathcal{L}_T$. In fact, if

$$(5.1) \quad P_1 = \alpha_0 I + \alpha_1 (P_1 + P_2) + \dots + \alpha_{n-1} P_n,$$

then

$$P_1 = P_1 P_1 = (\alpha_0 I + \alpha_1 (P_1 + P_2) + \dots + \alpha_{n-1} P_n) P_1 = (\alpha_0 + \alpha_1) P_1$$

which implies that $\alpha_0 + \alpha_1 = 1$. On the other hand, if we do the something, but this time with P_2 in (5.1), we get that $\alpha_0 + \alpha_1 = 0$, which is a contradiction. In similar way, we can prove that $P_2 \notin \mathcal{L}_T$.

Observe that

$$\Psi(\eta_{\{\lambda_k\}}) = G^{-1}(\eta_{\{\lambda_k\}}) = P_k, \quad k \neq 0$$

In fact, the Gelfand transformation of P_k is the following:

$$\begin{aligned} G_{P_k}(\phi_j) &= \phi_j(P_k) = \phi_j \left(\prod_{i=0, i \neq k}^n \left(\frac{\lambda_i I - T}{\lambda_i - \lambda_k} \right) \right) \\ &= \prod_{i=0, i \neq k}^n \left(\frac{\phi_j(\lambda_i I) - \phi_j(T)}{\lambda_i - \lambda_k} \right) = \prod_{i=0, i \neq k}^n \left(\frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_k} \right) \\ &= \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} = \eta_{\{\lambda_k\}}(\lambda_j) \end{aligned}$$

If $k = 0$,

$$\Psi(\eta_{\{\lambda_0\}}) = G^{-1}(\eta_{\{\lambda_0\}}) = I - \sum_{i=1}^n P_i$$

Note that the corresponding Boolean ring $\Omega(\sigma(T))$ is the collection of all subset of $\sigma(T)$, and its measure

$$m : \Omega(\sigma(T)) \rightarrow \mathcal{L}_T$$

is defined by $m(\{\lambda_k\}) = \Psi(\eta_{\{\lambda_k\}})$.

If the eigenvalues are pairwise different, then

$$\begin{aligned}
m(\emptyset) &= 0 \\
m(\sigma(T)) &= I \\
m(\{\lambda_0\}) &= I - \sum_{i=1}^n P_i; \\
m(\{\lambda_i\}) &= P_i \text{ for } i \neq 0 \\
m(\{\lambda_0, \lambda_i\}) &= I - \sum_{1 \leq j \leq n; j \neq i} P_j
\end{aligned}$$

and for $f \in C(\sigma(T), \mathbb{K})$

$$\int_{\sigma(T)} f dm = f(\lambda_0)I + \sum_{i=1}^n [f(\lambda_i) - f(\lambda_0)] P_i.$$

On the other hand, if a couple of the eigenvalues are equal, say for example $\lambda_1 = \lambda_2$ and the rest of them are different, then

$$\begin{aligned}
m(\emptyset) &= 0 \\
m(\sigma(T)) &= I \\
m(\{\lambda_0\}) &= I - (P_1 + P_2) - \sum_{i=3}^n P_i \\
m(\{\lambda_1\}) &= P_1 + P_2 \\
m(\sigma(T) - \{\lambda_1\}) &= I - (P_1 + P_2) \\
m(\{\lambda_i\}) &= P_i \text{ if } i \neq 1, 2 \\
m(\{\lambda_0, \lambda_1\}) &= I - \sum_{j=3}^n P_j \\
m(\{\lambda_0, \lambda_i\}) &= I - (P_1 + P_2) - \sum_{j=3}^n P_j \text{ if } i \neq 1, 2 \text{ and } j \neq i
\end{aligned}$$

and for $f \in C(\sigma(T), \mathbb{K})$

$$\int_{\sigma(T)} f dm = f(\lambda_0)I + [f(\lambda_1) - f(\lambda_0)](P_1 + P_2) + \sum_{i=3}^n [f(\lambda_i) - f(\lambda_0)] P_i.$$

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